Robot Kinematics
Lecture 3 (and maybe 4): Singularities

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Overview of the third lecture

Singularities: a classical topic for differentiable maps

Perturbation of a singularities: generically, only stable singularities

Screws: indispensable in mechanics, Lie algebra of the Lie group of rigid motions, useful for analyzing singularities
Singularities and Jacobian determinant of the IKM

▶ A solution \((\varphi_0, x_0, y_0)\) of the DKP can be continuously followed under small variation of \(\rho_1, \rho_2, \rho_3\) if the Jacobian matrix of the IKM: \((\varphi, x, y) \mapsto (\rho_1, \rho_2, \rho_3)\) is invertible at \((\varphi_0, x_0, y_0)\) (local inverse function theorem).

▶ Parallel singularities: poses such that the Jacobian determinant \(\text{Jac}(\varphi, x, y)\) vanishes. They form a surface in the work space. Image by IKM: surface in the joint space.

Singularities are dangerous for the robot!

Exercise:
Singular poses for the 3-RPR = the three limbs are concurrent. Pulling hard on one limb destroys the robot.
Computing the singularities of 3-RPR

- One forms the jacobian ideal of the 3-RPR: the ideal generated by the jacobian determinant of the IKM and the equations of the IKM (in variables $x, y, t, \rho_1, \rho_2, \rho_3$).

- We eliminate the pose variables $x, y, t$: compute the intersection of the jacobian ideal with $\mathbb{R}[\rho_1, \rho_2, \rho_3]$.

- We recover up to a parasitic factor the discriminant of the degree 6 equation in $t$. 

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Singularities of differentiable maps

- A differentiable map $f : M \to N$ (such as the IKM from Workspace to Joint space).
- We restrict to dimension 2: $M$ a neighborhood of $(0, 0)$ in $\mathbb{R}^2$, with coordinates $(x, y)$, $N$ a neighborhood of $(0, 0)$ in $\mathbb{R}^2$ with coordinates $(p, q)$, $f(x, y) = (p(x, y), q(x, y))$.
- $f(0, 0) = (0, 0)$ and $(0, 0)$ is a singular point of $f$:
  \[
  \text{Jac}(f)(0, 0) = \begin{vmatrix}
  \frac{\partial p}{\partial x}(0, 0) & \frac{\partial p}{\partial y}(0, 0) \\
  \frac{\partial q}{\partial x}(0, 0) & \frac{\partial q}{\partial y}(0, 0)
  \end{vmatrix} = 0
  \]
- **Theorem (Whitney):** There are two types of stable singularities (which remain the same under small perturbations of $f$): **fold** (codimension 1) and **cusp** (or pleat) (codimension 2). Stable singularities are dense in germs of plane maps.
Fold

- Normal form (up to a differentiable change of variables at the source and at the target): $\begin{cases} p = x \\ q = y^2 \end{cases}$
- Jacobian: $2y$.
- Image singularity curve (critical values): $q = 0$
- Codimension 1: curve in dimension 2, surface in dimension 3
- Double solution to $f(x, y) = (0, 0)$. Two solutions for $q > 0$, zero for $q < 0$
Views of a fold

- In the plane $(x, y)$

- In the plane $(p, q)$
Pleat (cusp)

- Normal form: \[ \begin{align*} p &= x \\ q &= y^3 + xy \end{align*} \]

- Jacobian: \( 3y^2 + x \). Smooth curve of singularities (parabola).

- Inmage singularity curve: \( 4p^3 + 27q^2 = 0 \).

- The cusp is at the origin: codimension 2. Point in dimension 2, curve in dimension 3.

- Triple solution of \( f(x, y) = (0, 0) \). One solution for \( 4p^3 + 27q^2 > 0 \), three for \( 4p^3 + 27q^2 < 0 \); one can go smoothly from the “lowest” solution to the “highest” one by circling around the cusp in the plane \((p, q)\).

- The cusp is a stationary point on the image singularity curve.
Views of a pleat (cusp)

- In the plane \((x, y)\)
- In the plane \((p, q)\)
Perturbation of an unstable singularity

Complex square map: \( f : (x, y) \mapsto (x^2 - y^2, 2xy) \). The jacobian matrix is the zero matrix at the origin. Singularity: only the origin.

Perturbation to \( f_\epsilon : (x, y) \mapsto (x^2 - y^2 + \epsilon x, 2xy - \epsilon y) \). A circle of singularities with three cusps appear:
2-RPR PR

2-dof planar parallel manipulator:

Workspace coordinates: $\varphi, y$.
Joint space coordinates: $\ell_1, \ell_2$ (or better their squares).
(\varphi = \pi, \ y = 0) is a solitary singularity. Looping around its image in joint space yields a non-singular assembly mode change. Second loop: get back the first assembly mode.
Perturbated 2-RPR PR

$B_2, B, B_1$ no longer aligned.
Three cusps in the deltoid. Looping around the deltoid has the same effect as looping around the solitary singularity in the non-generic case.
The return of the 6-UPS (Gough-Stewart)

IKM Gough-Stewart:

\[ \text{SE}(3) \rightarrow \mathbb{R}^6 \]

from the group of rigid motions to the space of lengths of limbs.

Differential of the IKM:

\[ \mathfrak{se}(3) \rightarrow \mathbb{R}^6 \]

\[ \vec{T}_{\text{platform}} \mapsto (\dot{r}_1, \ldots, \dot{r}_6) \]

\[ \vec{T}_{\text{platform}}: \text{field of velocities of the platform (twist)}, \]

\[ \dot{r}_i: \text{joint velocities}. \]
Screws, twists and wrenches

- **Screw:** vector field $\mathbf{T}$ on euclidean 3-space of the form $\mathbf{T}(M) = \mathbf{q} + \mathbf{p} \times \overrightarrow{OM}$. The reduct at $O$ of the screw is $(\mathbf{p}, \mathbf{q})$.

- **Twist:** field of velocities of a rigid body, $\mathbf{v}_M = \mathbf{v}_O + \mathbf{\omega} \times \overrightarrow{OM}$, where $\mathbf{\omega}$ is the angular velocity vector. Reduct at $O$: $(\mathbf{\omega}, \mathbf{v}_O)$. The twists form a 6-dimensional space, the Lie algebra $se(3)$. Matrix form: $\begin{pmatrix} A_{\mathbf{\omega}} & \mathbf{v}_O \\ 0 & 0 \end{pmatrix}$, where $A_{\mathbf{\omega}}$ is the skew-symmetric matrix of $\mathbf{\omega} \times \cdot$.

- **Wrench:** field of torques (moments) of a system of forces, $\mathbf{m}_M = \mathbf{m}_O + \mathbf{f} \times \overrightarrow{OM}$, where $\mathbf{f}$ is the resultant of the system of forces. Reduct at $O$: $(\mathbf{f}, \mathbf{m}_O)$.
Reciprocal product. Plücker coordinates

- **Reciprocal product** of two screws $\vec{T}_1$ and $\vec{T}_2$ with reducts $(\vec{p}_1, \vec{q}_1)$ and $(\vec{p}_2, \vec{q}_2)$: $\vec{T}_1 \circ \vec{T}_2 = \vec{p}_1 \cdot \vec{q}_2 + \vec{q}_1 \cdot \vec{p}_2$. Independent of the reduction point, nondegenerate symmetric bilinear form.

- Reciprocal product of a wrench and a twist = power produced.

- Reciprocal screws: orthogonal w.r.t. reciprocal product.

- **Plücker coordinates** of a line: reduct of a pure force along this line (homogeneous coordinates). For a line $(AB)$: $\mathcal{P}_{A,B} = (\vec{AB}, \vec{OA} \times \vec{AB})$. Plücker coordinates for the line at infinity of a plane orthogonal to $\vec{n}$: $(\vec{0}, \vec{n})$.

- Plücker coordinates of lines are the nonzero self-reciprocal screws. Lines are intersecting (or parallel) iff their Plücker coordinates are reciprocal.
Screw analysis of the singularities

- Decomposition of the twist of the platform along a leg $A_iB_i$, according to joints (composition of velocities):
  $$\vec{T}_{\text{platform}} = \vec{T}_{U_i} + \vec{T}_{P_i} + \vec{T}_{S_i}.$$  

- Reciprocal product with $\mathcal{P}_{A_i,B_i}$: no power produced at $U_i$ and $S_i$, so
  $$\mathcal{P}_{A_i,B_i} \odot \vec{T}_{\text{platform}} = \mathcal{P}_{A_i,B_i} \odot \vec{T}_{P_i} = r_i \dot{r}_i .$$

- A pose is singular iff the Plücker coordinates of the legs are linearly dependent.